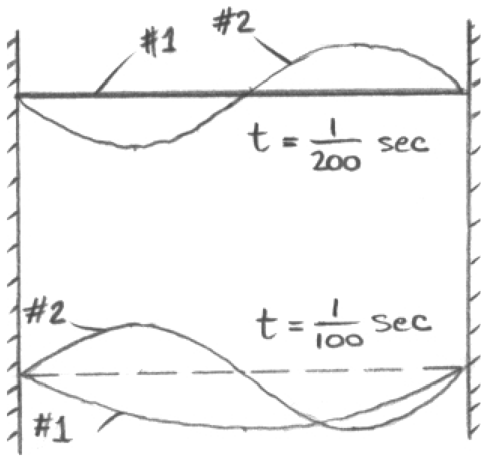
University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 11

Composed and formatted by E.A. Baltz and M. Strovink; proofed by D. Bacon

1. French 6-9.

(a.) The lowest resonant frequency of a room is 50 Hz. All integer multiples of this frequency are also resonant. The lowest two modes are excited. These are 50 Hz (the fundamental) and 100 Hz (the [first] harmonic). The amplitude is maximum at $t = 0$. The time interval $t = 1/200$ sec is one fourth of a period for the fundamental and half a period for the harmonic. $t = 1/100 sec$ is one half of a period for the fundamental and a full period for the harmonic. These modes look like



(b.) We can write the total displacement as

$$\xi(x) = A_1 \sin(\pi x/L)\cos(100\pi t) + \\ + A_2 \sin(2\pi x/L)\cos(200\pi t)$$

where A_1 and A_2 are the unknown amplitudes for the fundamental and harmonic mode, respectively. In particular

$$\xi(L/2) = A_1 \cos(100\pi t) + 0$$
$$\xi(L/4) = \frac{A_1}{\sqrt{2}} \cos(100\pi t) + A_2 \cos(200\pi t)$$
$$\xi(3L/4) = \frac{A_1}{\sqrt{2}} \cos(100\pi t) - A_2 \cos(200\pi t)$$

Since the amplitude at $L/2$ is due only to the fundamental, and is equal to 10 μ, we know that $A_1 = 10\ \mu$. The last two equations become

$$\xi(L/4) = 5\sqrt{2}\cos(100\pi t) + A_2 \cos(200\pi t)$$
$$\xi(3L/4) = 5\sqrt{2}\cos(100\pi t) - A_2 \cos(200\pi t)$$

As a trial solution, we assume that A_2 is positive. Then the maximum displacement of 10 μ at $L/4$ is reached at $t = 0$; therefore $A_2 = (10 - 5\sqrt{2})$ μ. The maximum |displacement| at $3L/4$, a negative displacement in this case, is reached at $t = 1/100$ sec, when the harmonic has changed phase by a full 2π, but the fundamental has changed phase by only π and has therefore become negative. With the above values of A_1 and A_2, the displacement at $3L/4$ is equal to -10 μ, in agreement with the problem.

2. French 6-15(a).

This is a bit messy, but bear with it. The string has length L. Its initial conditions are $y(x, 0) = Ax(L - x)$ and $(\partial y/\partial t)_{t=0} = 0$. We write the solution as a Fourier series

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

We know the solution at $t = 0$. Define $B_n = A_n \cos\delta_n$.

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

We can now solve for the B_n using Fourier's trick:

$$B_n = \frac{2}{L}\int_0^L Ax(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Integrating by parts once, notice that the surface term vanishes:

$$B_n = \frac{2A}{n\pi}\int_0^L (L-2x)\cos\left(\frac{n\pi x}{L}\right)dx$$

The L term integrates now. It always integrates to zero.

$$B_n = -\frac{4A}{n\pi}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

We integrate by parts again, and again the surface term vanishes.

$$B_n = \frac{4AL}{n^2\pi^2}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$

This integrates easily.

$$B_n = -\frac{4AL^2}{n^3\pi^3}\cos\left(\frac{n\pi x}{L}\right)_0^L = \left.\frac{8AL^2}{n^3\pi^3}\right|_{n\ \mathrm{odd}}$$

We see that B_n is zero for all even n. We expected this because the initial condition is symmetric around the center of the string. Now we tackle the velocities.

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\omega_n \sin(\delta_n) = 0$$

This equation is satisfied if we simply set all of the $\delta_n = 0$, so that $\sin\delta_n = 0$ and $\cos\delta_n = 1$. This also means that $A_n = B_n$. We now have the full solution. We have rewritten the sum to only include odd n.

$$y(x,t) = \sum_{m=0}^{\infty}\frac{8AL^2}{(2m+1)^3\pi^3}\times$$
$$\times \sin\left(\frac{(2m+1)\pi x}{L}\right)\cos(\omega_{2m+1}t)$$

3. French 8-9.
This problem concerns a very important effect called Doppler broadening. Sodium atoms emit light of 6000Å. The observed light varies in a small frequency range of (6000 ± 0.02)Å. This is caused by the thermal motion of the sodium atoms. The Doppler effect tells us that

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{0.02}{6000} = 3.33\times 10^{-6}$$

This gives the maximum velocity of the atoms $v_{\max} = 1000$ m/sec. The thermal velocity is given by

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT \Rightarrow T = \frac{m\langle v^2\rangle}{3k}$$

The constant $k = 1.38\times 10^{-23}$ J/Kelvin is Boltzmann's constant, and the appropriate mass $m \approx 23m_p$ is just the mass of the sodium atom. Approximating $\sqrt{\langle v^2\rangle} \approx v_{\max}$, this gives $T \approx 900$ Kelvin. This effect can be used to measure the temperatures of objects in astronomy.

4. French 8-12.
The Doppler effect for a moving source and fixed observer is given by

$$\nu(\theta) = \frac{\nu_0}{1 - \frac{u}{v}\cos\theta}$$

(a.) We want to find the Doppler effect for a fixed source and a moving observer at velocity $-u$. We are going to do this differently than it was done in French, in order to provide an alternative and perhaps easier way to think about it. We are going to go into the rest frame of the wave crests, so the only thing moving will be the observer and the source. The velocities of the wave and the observer are

$$V_w = (v, 0) \qquad V_o = (-u\cos\theta, u\sin\theta)$$

Transforming into the rest frame of the wave by subtracting $(v, 0)$, the velocity of the observer is

$$V_o = (-v - u\cos\theta, u\sin\theta)$$

The distance between wave crests is just $L = v/\nu_0$, so the rate at which the observer crosses the wave crests is just the x velocity divided by this distance. This is the shifted frequency.

$$\nu'(\theta) = \nu_0\left(1 + \frac{u}{v}\cos\theta\right)$$

This trick doesn't work for light because light doesn't have a rest frame. In relativity, there is a different Doppler formula that applies to both situations. It uses only the relative velocity of the source and the observer.

(b.) We want to know the approximate difference between the two formulas. We Taylor expand the first formula, assuming that the speed is much less than the sound speed, $u \ll v$.

$$\nu(\theta) \approx \nu_0 \left(1 + \frac{u}{v}\cos\theta + \frac{u^2}{v^2}\cos^2\theta + \cdots\right)$$

This tells us the approximate difference between the two Doppler shifted frequencies

$$\nu(\theta) - \nu'(\theta) \approx \nu_0 \frac{u^2}{v^2}\cos^2\theta$$

5. We can use Bernoulli's equation to find the velocities of these two flows.

(a.) If we look right as the flow is leaving the tank, we see that it must be at atmospheric pressure. The stream is arbitrarily thin, so this is the only possibility. We then just use the gravitational potential to find the velocities. If the hole were at the top of the tank, the velocity would be zero, so we use this as the constant.

$$P_0 + \rho g(H-h) + \frac{1}{2}\rho v^2 = P_0 + \rho g H$$
$$v(h) = \sqrt{2gh}$$

The time it takes to hit the ground is given by the formula for constant acceleration.

$$H - h = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2(H-h)}{g}}$$

In this time, the water travels a horizontal distance vt, so the distance from the tank is

$$d(h) = \sqrt{4h(H-h)}$$

(b.) We notice that the previous formula says that water leaking from a depth h travels the same distance as water leaking from a depth $H-h$.

6. We again apply Bernoulli's equation. The air at the leading edge is stagnant, and we will assume that it is at atmospheric pressure. Bernoulli's equation gives

$$P_0 = P + \frac{1}{2}\rho v^2$$

The maximum velocity occurs when the pressure is zero. Plugging in P_0=10^5 N/m^2 and $\rho = 1.2$ kg/m^3, we find that v =408 m/sec. This is larger than the speed of sound, $v_s = 340$ m/sec.

7. The gradient ∇f is equal to

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

The x component of $\nabla \times (\nabla f)$ is the difference between the y and z partial derivatives, respectively, of the z and y components of ∇f:

$$(\nabla \times (\nabla f))_x = \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}$$

Interchanging the order of differentiation in either of the terms, this expression is seen to vanish for well-behaved f. By cyclic permutation, the y and z components of $\nabla \times (\nabla f)$ vanish as well.

8. (a.) When the point of observation (x, y, z) is displaced incrementally by $d\mathbf{s}$, where

$$d\mathbf{s} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$$

points in an arbitrary direction, the change df in $f(x, y, z)$ is given by the chain rule:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

The right-hand side can be rewritten as the dot product of

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

and $d\mathbf{s}$ above:

$$df = \nabla f \cdot d\mathbf{s}$$

For a fixed length $|d\mathbf{s}|$, this dot product is greatest when $d\mathbf{s}$ is parallel to ∇f. Therefore, when $df/|d\mathbf{s}|$ is a maximum, the direction of $d\mathbf{s}$ will be along ∇f. With $f = x^2 + y^2 - z^2$, this direction $\hat{\mathbf{n}}$ is

$$\begin{aligned}\hat{\mathbf{n}} &= \frac{\hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}}{|\nabla f|} \\ &= \frac{\hat{\mathbf{x}}2x + \hat{\mathbf{y}}2y - \hat{\mathbf{z}}2z}{2\sqrt{x^2+y^2+z^2}} \\ &= \frac{\hat{\mathbf{x}}6 + \hat{\mathbf{y}}8 - \hat{\mathbf{z}}10}{2\sqrt{9+16+25}} \\ &= \frac{\hat{\mathbf{x}}3 + \hat{\mathbf{y}}4 - \hat{\mathbf{z}}5}{5\sqrt{2}}\end{aligned}$$

(b.) The surface $z(x, y) = \sqrt{x^2 + y^2}$ can be described as

$$0 = f(x, y, z) = x^2 + y^2 - z^2$$

This is the same $f(x, y, z)$ as in part (a.). Suppose the point of observation (x, y, z) is displaced infinitesimally by $d\mathbf{v}$, where $d\mathbf{v}$ is *on the surface* $f = 0$. Then we would expect f not to change at all. However, according to the results of part (a.),

$$df = \nabla f \cdot d\mathbf{v}$$

Therefore df can vanish only if $d\mathbf{v}$ is perpendicular to ∇f. Since $d\mathbf{v}$ can be any displacement which lies on the surface, this requires ∇f to be perpendicular to the surface. Therefore, the direction of the normal to the surface $d\mathbf{u}$ in part (b.) is the same as the direction $\hat{\mathbf{n}}$ of $d\mathbf{s}$ in part (a.), the direction of maximum change in f.

9. The fluid velocity field is

$$\mathbf{v}(x, y, z, t) = (\hat{\mathbf{y}}x - \hat{\mathbf{x}}y)\omega(t)$$

(a.) The equation of continuity (conservation of fluid molecules) requires

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Therefore

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\omega \nabla \cdot (\hat{\mathbf{y}}x\rho - \hat{\mathbf{x}}y\rho) \\ \frac{1}{\omega}\frac{\partial \rho}{\partial t} &= y\frac{\partial \rho}{\partial x} - x\frac{\partial \rho}{\partial y}\end{aligned}$$

(b.) An element of fluid at $\mathbf{r}_\perp = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ has a velocity $\mathbf{v} = (\hat{\mathbf{y}}x - \hat{\mathbf{x}}y)\omega(t)$ that is always in the xy plane and orthogonal to $\mathbf{r}_\perp$. Thus the element is in circular motion about the z axis (to which $\mathbf{r}_\perp$ is perpendicular), with angular velocity

$$\boldsymbol{\Omega} = \hat{\mathbf{z}}\frac{|\mathbf{v}|}{|\mathbf{r}_\perp|} = \hat{\mathbf{z}}\omega(t)$$

On the other hand

$$\begin{aligned}\nabla \times \mathbf{v} &= \omega(t)\,\hat{\mathbf{z}}\Big(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\Big) \\ &= 2\hat{\mathbf{z}}\omega(t)\end{aligned}$$

(c.) Suppose that the independent variables (x, y, z, t) upon which a vector $\mathbf{A}(x, y, z, t)$ depends change infinitesimally, by (dx, dy, dz, dt). Then, by the chain rule, a component of $\mathbf{A}$, *e.g.* A_x, changes by an amount

$$\begin{aligned}dA_x &= \frac{\partial A_x}{\partial t}dt + \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz \\ \frac{dA_z}{dt} &= \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z \\ &= \frac{\partial A_x}{\partial t} + \Big(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\Big)A_x \\ \frac{dA_x}{dt} &= \Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)A_x \\ \Rightarrow \frac{d\mathbf{A}}{dt} &= \Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{A}\end{aligned}$$

(This is the *convective derivative*, yielding the total time rate of change of $\mathbf{A}$.) In this problem $\mathbf{A} = \mathbf{v}$ itself, so

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{v} \\ &= 0 + \omega_0\Big(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y}\Big)(\hat{\mathbf{y}}x - \hat{\mathbf{x}}y) \\ &= \omega_0^2\Big(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\Big)(\hat{\mathbf{y}}x - \hat{\mathbf{x}}y) \\ &= \omega_0^2(-y\hat{\mathbf{y}} - x\hat{\mathbf{x}}) \\ &= -\omega_0^2\mathbf{r}_\perp\end{aligned}$$